Biom. J. 31 (1989) 2, 213–218

A Note on Block Designs with Nested Rows and Columns

MURARI SINGH

Indian Agricultural Statistics Research Institute, New Delhi and
International Crops Research Institute for the Semi-Arid Tropics, Patancheru

Summary

Block designs with nested rows and columns have been introduced by SINGH and DEY (1979). This note gives some results regarding the total relative loss of information and patterns of efficiency balanced block designs with nested rows and columns in variable replications of treatments.

Key words: Blocks with nested rows and columns; Variable replication; Efficiency balance; Information matrix.

1. Introduction

Block designs with nested rows and columns have been introduced by SINGH and DEY (1979). These designs are a set of incomplete blocks within which are nested two more classifications, rows and columns and are useful for the situations where it is desired to eliminate heterogeneity in two directions within each block. They have given the analysis and construction of several series of balanced designs.

Consider a block design with nested rows and columns in v treatments and in s blocks, each block containing pq treatments arranged in p rows and q columns. Let $\underset{\sim}{N}$ be the incidence matrix of treatments *vs* blocks. Further let $\underset{\sim}{N}_{1j}$ and $\underset{\sim}{N}_{2j}$ be $v\times p$ and $v\times q$ incidence matrices of treatments *vs* rows and treatments *vs* columns respectively, in the j-*th* block ($j=1, 2, \ldots, s$). Define

$$\underset{\sim}{N}_1=(\underset{\sim}{N}_{11}:\underset{\sim}{N}_{12}: \ldots : \underset{\sim}{N}_{1s})$$
$$\underset{\sim}{N}_2=(\underset{\sim}{N}_{21}:\underset{\sim}{N}_{22}: \ldots : \underset{\sim}{N}_{2s})$$

Let $\underline{r}$ be the $v\times 1$ column vector of treatment replications. It is easy to notice $\underset{\sim}{N}\underline{J}=\underset{\sim}{N}_1\underline{J}=\underset{\sim}{N}_2\underline{J}=\underline{r}$, where $\underline{J}$ stands for conformal column vector of all unities. Denote by $\underset{\sim}{r}^{\delta}$, the diagonal matrix with the elements of vector $\underline{r}$ in diagonal, and $\underset{\sim}{r}^{-\delta}$ the inverse of $\underset{\sim}{r}^{\delta}$.

The normal equations for estimating the effects of treatments after eliminating the effects of rows, columns and blocks are given by

$$\underset{\sim}{F}\underline{\tau}=\underline{Q} \tag{1.1}$$

where

$$\underset{\sim}{F} = \underset{\sim}{r}^{\delta} - \underset{\sim}{N}_1 \underset{\sim}{N}_1'/q - \underset{\sim}{N}_2\underset{\sim}{N}_2'/p + \underset{\sim}{N}\underset{\sim}{N}'/pq$$

and $\underset{\sim}{\tau}$ is $v \times 1$ column vector of treatment effects. The expression for $\underset{\sim}{Q}$ (the vector of adjusted treatment totals) can be seen in SINGH and DEY (1979) and is not produced here as it will not appear onward.

KSHIRSAGAR (1957) has obtained the expression for total loss of information for block designs eliminating heterogeneity in one direction and in two directions. The total relative loss of information for block designs with nested rows and columns is given in Section 2.

The balanced designs usually require a number of restrictions on the parameters, for instance, most of the known balanced designs are available with equal replications. Section 3 gives a result for obtaining efficiency balanced designs in variable replications of treatments.

2. Total Relative Loss of Information

The expressions for the total relative loss of information in block designs and row and column designs have been given by KSHIRSAGAR (1957) in terms of the trace of the information matrix of treatment effects obtained after eliminating the effects of other classifications. In this section, we obtain the expression for the total loss of information for *binary* (if each of the v treatments occurs at most once in each block) and *equireplicate* (if every treatment occurs in exactly r blocks) block designs with nested rows and columns.

To understand more precisely the various notations introduced, consider for example the following design given by SINGH and DEY (1979) in $v=5$ treatments and $s=5$ blocks of size $p=2$ rows and $q=2$ columns.

Blocks

B_1		B_2		B_3		B_4		B_5	
1	2	1	5	1	4	1	3	2	3
3	4	2	3	5	2	4	5	5	4

The various incidence matrices $\underset{\sim}{N}_{ij}$, $\underset{\sim}{N}_i$ ($i=1, 2; j=1, 2, 3, 4, 5$) are:

$$\underset{\sim}{N}_1 = (\underset{\sim}{N}_{11} : \underset{\sim}{N}_{12} : \underset{\sim}{N}_{13} : \underset{\sim}{N}_{14} : \underset{\sim}{N}_{15})$$

$$= \begin{array}{c} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \\ \text{Rows} \end{array} \begin{array}{c} \text{Treatments} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \end{array}$$

$$N_2 = (N_{21} : N_{22} : N_{23} : N_{24} : N_{25})$$

$$\begin{array}{c c}
 & \text{Treatments} \\
= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \\
\text{Columns} &
\end{array}$$

$$\begin{array}{c c}
 & \text{Treatments} \\
N = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \\
\text{Blocks} &
\end{array}$$

The sums of rows in each of the matrices N_1, N_2 and N yield the same replication vector given by

$$r = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

Further the matri

$$N_1N_1' = \begin{bmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 4 & 2 & 0 & 1 \\ 1 & 2 & 4 & 1 & 0 \\ 1 & 0 & 1 & 4 & 2 \\ 1 & 1 & 0 & 2 & 4 \end{bmatrix}, \quad N_2N_2' = \begin{bmatrix} 4 & 1 & 1 & 1 & 1 \\ 1 & 4 & 0 & 2 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 2 & 1 & 4 & 0 \\ 1 & 1 & 2 & 0 & 4 \end{bmatrix}, \quad NN' = \begin{bmatrix} 4 & 3 & 3 & 3 & 3 \\ 3 & 4 & 3 & 3 & 3 \\ 3 & 3 & 4 & 3 & 3 \\ 3 & 3 & 3 & 4 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{bmatrix}$$

Thus the matrix F in (1.1) is

$$F = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} - (1/2)\, N_1N_1' - (1/2)\, N_2N_2' + (1/4)\, NN'$$

$$= \begin{bmatrix} 1 & -1/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 1 & -1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1 & -1/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 1 & -1/4 \\ -1/4 & -1/4 & -1/4 & -1/4 & 1 \end{bmatrix} = 5/4\ I_4 - (1/4)\ J$$

$$= \ (5/4)\ (I - J/5).$$

Thus for equireplicate design, the F matrix is

$$(2.1)\qquad \underset{\sim}{F}=r\underset{\sim}{I}-\underset{\sim}{N}_1\underset{\sim}{N}_1'/q-\underset{\sim}{N}_2\underset{\sim}{N}_2'/p+\underset{\sim}{N}\underset{\sim}{N}'/pq$$

when the design is binary also, we get the trace of $\underset{\sim}{F}$ matrix as

$$\text{tr } \underset{\sim}{F}=rv\ (1-1/p-1/q+1/pq)$$

Let the rank of matrix $\underset{\sim}{F}$ be $v-1-\alpha,\ \alpha\geqq 0$. The design is said to be *connected* if $\alpha=0$. Using the procedure given by KSHIRSAGAR (1967), the total relative loss of information, say L, is

$$\begin{aligned} L&=v-1-\alpha-(1/r)\text{ tr } \underset{\sim}{F}\\ &=\frac{s\ (p+q-1)}{r}-1-\alpha \end{aligned}$$

since $vr=spq$, where v is the number of treatments. The above result can be stated in the following theorem.

Theorem 2.1. For the binary, equireplicate and connected block design with nested rows and columns, the total relative loss of information is $(s\,(p+q-1)/r)-1$, where s is the number of blocks, each having p rows and q columns and r is the common replication.

3. Balanced Designs With Variable Replications

We shall need the following definitions of efficiency-balanced and variance-balanced block design with nested rows and columns, (see, SINGH and DEY, 1979).

Definition 1. A block design with nested rows and columns is called efficiency-balanced block design with nested rows and columns (EBB-RC) if every contrast among treatment effects is estimated with same efficiency factor (efficiency-factor of a given contrast of treatment effects being the ratio of the variances of the best linear unbiased estimate of the contrast under an orthogonal design to the design under consideration).

A necessary and sufficient condition for a design to be EBB-RC is

$$(3.1)\qquad \underset{\sim}{M}o=\mu\ (\underset{\sim}{I}-\underset{\sim}{J}\underset{\sim}{r}'/n)$$

where $\underset{\sim}{M}o=\underset{\sim}{r}^{-\delta}\ (\underset{\sim}{N}_1'\underset{\sim}{N}_1'/q+\underset{\sim}{N}_2\underset{\sim}{N}_2'/p-\underset{\sim}{N}\underset{\sim}{N}'/pq-\underset{\sim}{r}\underset{\sim}{r}'/n)$, μ is the unique non-zero eigen value of $\underset{\sim}{M}o$ and $n=\underline{J}'\underline{r}$, the total number of observations.

Definition 2. A block design with nested rows and columns is called variance balanced block design with nested rows and columns (VBB-RC) if the variance of every elementary contrast of treatment effects estimated through the design is same.

A necessary and sufficient condition for a design to be VBB-RC is

$$(3.2)\qquad \underset{\sim}{F}=\Theta\ (\underset{\sim}{I}-\underset{\sim}{J}\underset{\sim}{J}'/v)$$

where Θ is the unique non-zero eigen value of matrix $\underset{\sim}{F}$ definited in (1.1).

In the theorem 3.1., we have an inter-relation between EBB-RC and VBB-RC designs.

Theorem 3.1. For a block design with nested rows and columns, any two of the following three conditions implies the third.

(*i*) The design is EBB-RC.
(*ii*) The design is VBB-RC.
(*iii*) The design is equireplicate.

The proof turns out to be simple in view of equations (3.1), (3.2) and $\underline{r}=r\underline{J}$, being the common replication.

The balanced designs with equireplicated treatments are of lesser practical utility than those designs which are balanced and permits the arbitrary replications. In the following we give some balanced designs with variable replications in m $(<v)$ treatments obtained by merging some of the v treatments of a balanced designs.

Consider a block design D with nested rows and columns in v treatments grouped into m disjoint classes, the i-th class having v_i treatments, say $(ai_1, ai_2, \ldots, a_{iv_i})$, $i=1, 2, \ldots, m$. Obtain a design D^* by replacing (merging) every treatment of i-th class by the same treatment, say a_i^*. Thus D^* is a block design with nested rows and columns in m treatments. We prove the following theorems.

Theorem 3.2. If there exists an EBB-RC design D in v treatments then the design D^* obtained in above manner in m $(<v)$ treatments is also an EBB-RC design.

Proof. Let $\underset{\sim}{N}$, $\underset{\sim}{N}_{1j}$, $\underset{\sim}{N}_{2j}$ $(j=1, 2, \ldots, s)$ be the various incidence matrices of design D and $\underline{r}$ be the replication vector for v treatments. Denite by $P=P(\underline{J}'_{v1}, \ldots, \underline{J}_{vn})$. $m\times v$ matrix with $1\times v_i$ row vectors $\underline{J}'_{vj}$ of all unities in i-th diagonal positions of an identity matrix of order $m\times m$, $i=1, 2, \ldots, m$. It is now easy to notce that the design D^* obtained by merging v treatments of D into m-treatments of D^* has the following incidence matrices, $\underset{\sim}{N}^*$, $\underset{\sim}{N}_1^*$, $\underset{\sim}{N}_2^*$

$$\begin{aligned} &\underset{\sim}{N}^*=\underset{\sim}{P}\ \underset{\sim}{N} \\ (3.3)\qquad &\underset{\sim}{N}_1^*=\underset{\sim}{P}\ \underset{\sim}{N}_1 \\ &\underset{\sim}{N}_2^*=\underset{\sim}{P}\ \underset{\sim}{N}_1 \end{aligned}$$

The replication vector $\underline{r}^*$ is given $\underline{r}^*=\underset{\sim}{P}\underline{r}$

Since D is an EBB-RC design, we have

$$(3.4)\qquad \underset{\sim}{M}o=(\underset{\sim}{I}-\underline{J}\underline{r}'/n)$$

The $\underset{\sim}{M}o$-matrix for D^* denoted by $\underset{\sim}{M}o^*$ is

$$(3.5)\qquad \underset{\sim}{M}_0^*=\underset{\sim}{r}^{*-\delta}[\underset{\sim}{N}_1^*\underset{\sim}{N}_1^{*\prime}/q+\underset{\sim}{N}_2^*\underset{\sim}{N}_2^{*\prime}/p-\underset{\sim}{N}^*\ \underset{\sim}{N}^{*\prime}/pq-\underline{r}^*\underline{r}^{*\prime}/n]$$

Using (3.3) and (3.4), (3.5) simplifies to

$$\underset{\sim}{M}_0^*=(\underset{\sim}{I}-\underline{J}\underline{r}^{*\prime}/n)$$

This completes the theorem.

Several series of equireplicate VBB-RC designs have been constructed by SINGH and DEY (1979). Using theorem 3.1. and 3.2., we can easily prove the following result.

Theorem 3.3. Let D be an equireplicate VBB-RC design with Θ as the unique non-zero eigenvalue of the matrix $\underset{\sim}{F}$ given in (1.1.), then the design D^* obtained by merging the treatments in the manner described in Theorem 3.2., is EBB-RC design with $\mu=(r^*-\Theta)/r$ as the unique non-zero eigenvalue of $\underset{\sim}{Mo}$ given in (3.1.).

The application of Theorem 3.3. can be made in obtaining some EBB-RC design with variable replications as follows:

1. Plan of EBB-RC ($v=3$, $s=5$, $\underline{r}'=(4, 8, 8)$, $p=2$, $q=2$, $\mu=5/16$)

$$\begin{matrix}1&2\\3&2\end{matrix}\quad\begin{matrix}1&3\\2&3\end{matrix}\quad\begin{matrix}1&2\\3&2\end{matrix}\quad\begin{matrix}1&3\\2&3\end{matrix}\quad\begin{matrix}2&2\\3&3\end{matrix}$$

2. Plan of EBB-RC ($v=6$, $s=14$, $\underline{r}'=(7, 7, 7, 7, 14, 14)$, $p=2$, $q=2$, $\mu=2/7$)

$$\begin{matrix}1&2\\3&4\end{matrix},\quad\begin{matrix}1&5\\2&6\end{matrix},\quad\begin{matrix}1&3\\5&5\end{matrix},\quad\begin{matrix}1&5\\6&2\end{matrix},\quad\begin{matrix}1&6\\6&3\end{matrix},\quad\begin{matrix}1&4\\5&6\end{matrix}$$

$$\begin{matrix}1&6\\4&5\end{matrix},\quad\begin{matrix}2&6\\3&5\end{matrix},\quad\begin{matrix}2&3\\5&6\end{matrix},\quad\begin{matrix}2&4\\6&6\end{matrix},\quad\begin{matrix}2&5\\5&4\end{matrix},\quad\begin{matrix}3&5\\4&6\end{matrix},\quad\begin{matrix}3&6\\5&4\end{matrix}$$

$$\begin{matrix}5&6\\5&6\end{matrix}$$

3. Plan of EBB-RC ($v=8$, $s=18$, $\underline{r}'=(8, 8, 8, 8, 8, 8, 8, 16)$, $p=2$, $q=2$, $\mu=2/7$)

$$\begin{matrix}1&5\\7&2\end{matrix},\quad\begin{matrix}3&4\\7&6\end{matrix},\quad\begin{matrix}6&1\\3&7\end{matrix},\quad\begin{matrix}2&0\\4&5\end{matrix},\quad\begin{matrix}5&6\\2&3\end{matrix},\quad\begin{matrix}7&7\\0&1\end{matrix},\quad\begin{matrix}2&3\\7&0\end{matrix},$$

$$\begin{matrix}4&5\\6&7\end{matrix},\quad\begin{matrix}0&4\\6&1\end{matrix},\quad\begin{matrix}5&3\\7&7\end{matrix},\quad\begin{matrix}4&7\\1&5\end{matrix},\quad\begin{matrix}6&7\\2&0\end{matrix},\quad\begin{matrix}7&2\\4&7\end{matrix},\quad\begin{matrix}0&1\\5&3\end{matrix},\quad\begin{matrix}7&0\\5&6\end{matrix}$$

$$\begin{matrix}1&2\\3&4\end{matrix},\quad\begin{matrix}3&7\\0&4\end{matrix},\quad\begin{matrix}7&6\\1&2\end{matrix}$$

References

SINGH, M. and DEY, A., 1979: Block designs with nested rows and columns. Biometrika 66, 321–326.

KSHIRSAGAR, A. M., 1957: A note on the total relative loss of information in any design. Cal. Stat. Assoc. Bull. 7, 78–81.

Received: Nov. 1987
Revised: Jan. 1988

Dr. Murari SINGH
ICRISAT Patancheru P. O.
Andhra Pradesh 502324
India