Statistics & Probability Letters 5 (1987) 421–424
North-Holland October 1987

ROBUSTNESS OF ROW-COLUMN DESIGNS

Gulab SINGH
Planning Commission, New Delhi, India

Sudhir GUPTA
Northern Illinois University, DeKalb, IL 60115, USA

Murari SINGH
ICRISAT, Patancheru, Andra Pradesh, India

Received October 1986
Revised January 1987

Abstract: Robustness of designs eliminating heterogeneity in two directions to outliers is studied. The important class of variance balanced row-column designs which satisfy the property of adjusted orthogonality are shown to be robust. Some general three-way balanced designs with balanced column vs row classification are also shown to be robust.

Keywords: variance balance, three-way design, adjusted orthogonality, outlier, robustness, row–column design.

1. Introduction

Box and Draper (1975) studied robustness of response surface designs to outliers. For the full rank linear model $E(Y) = X\beta$, $D(Y) = \sigma^2 I$, they showed that for the predicted response at any point to be insensitive to the outlier, $r = \Sigma_u r_{uu}^2$ should be minimized, where r_{uu} is the uth diagonal element of the matrix $R = X(X'X)^{-1}X$. Gopalan and Dey (1976) extended their criterion to other experimental designs, and studied the robustness of some important classes of block designs. They pointed out that where the model is not of full rank, all the diagonal elements of $X(X'X)^{-}X$ should be equal for the robustness of designs. Here $(X'X)^{-}$ denotes a generalized inverse of $X'X$.

The purpose of this paper is to study the robustness of row-column designs. In the class of variance balanced designs with orthogonal row-column classification, it is shown that designs which satisfy the property of adjusted orthogonality of Eccleston and Russel (1975, 1977) are robust. Robustness of variance balanced designs with non-orthogonal variance balanced column vs row classification is also considered. A new class of such designs which are robust is also given.

2. Model and the variance of the predicted response

We consider the usual additive model for row-column designs which may be written as,

$$Y_{ijl} = \mu + \rho_i + \gamma_j + \tau_l + \varepsilon_{ijl}$$

where μ is the general mean, ρ_i, γ_j and τ_l are the effects of the ith row, jth column and lth treatment respectively, and ε_{ijl} are independent random errors with $E(\varepsilon_{ijl}) = 0$ and $E(\varepsilon_{ijl}^2) = \sigma^2$. In matrix notation the above model will be written as $Y = X\beta + \varepsilon$.

Let $S = X(X'X)^- X$ and let s_i be its ith diagonal element. As mentioned above, an experimental design will be robust if all s_i are equal. Observe that the dispersion matrix of the predicted response $\hat{Y} = X\hat{\beta}$ is given by

$$D(\hat{Y}) = X(X'X)^- X\sigma^2 = S\sigma^2.$$

The condition of robustness that all diagonal elements of S be equal, therefore amounts to the equality of the variances of the predicted responses $\hat{Y}_{ijl} = \hat{\mu} + \hat{\rho}_i + \hat{\gamma}_j + \hat{\tau}_l$. Thus a row-column design will be robust if and only if $\operatorname{Var}(\hat{Y}_{ijl})$ is constant independent of i, j and l.

Let N_1, N_2 and W denote respectively the $v \times p$ treatment-row, the $v \times q$ treatment-column and the $p \times q$ row-column incidence matrices where v, p and q denotes the numbers of treatments, rows and columns respectively. A general set-up is assumed in which the row-column classification may be non-orthogonal with k, h being the constant numbers of units in the rows and columns respectively, and r denotes the constant number of treatment replications. Let R, C, Q, ρ, γ and τ denote the column vectors of row totals, column totals, adjusted treatment totals, row effects, column effects and treatment effects respectively, and 1 denote a column vector of 1's of appropriate order. Let,

$$\begin{aligned} A &= rI - N_1N_1'/k, & B' &= N_2 - N_1W/k, \\ Z &= hI - W'W/k, & F &= A - B'Z^-B. \end{aligned} \tag{2.1}$$

Then, following Agrawal (1966) we have

$$\hat{\mu}1 + \hat{\rho} = \frac{1}{k}\{R - WZ^-(C - W'R/k - B\hat{\tau}) - N_1'\hat{\tau}\}, \tag{2.2}$$

$$\hat{\gamma} = Z^-(C - W'R/k - B\hat{\tau}), \qquad \hat{\tau} = F^-Q. \tag{2.3}$$

The variance of the estimated response $\hat{Y}_{ijl}$ is then obtained using the relation,

$$\begin{aligned} V(\hat{Y}_{ijl}) = {} & V(\hat{\mu} + \hat{\rho}_i) + V(\hat{\gamma}_j) + V(\hat{\tau}_l) + 2\operatorname{Cov}(\hat{\mu} + \hat{\rho}_k, \hat{\gamma}_j) \\ & + 2\operatorname{Cov}(\hat{\mu} + \hat{\rho}_i, \hat{\tau}_l) + 2\operatorname{Cov}(\hat{\gamma}_j, \hat{\tau}_l). \end{aligned} \tag{2.4}$$

3. Designs with orthogonal row-column classification

For row-column designs with orthogonal row-column classification $W = 11'$, and the various dispersion matrices $D(\cdot)$ for this case other than the common factor σ^2, are conveniently presented in Table 1. The variance of the estimated response $\hat{Y}_{ijl}$ given by (2.4) can then be obtained using Table 1. The design will be robust if the variance is constant for all possible values of i, j and l.

General conclusions are possible in the class of variance balanced designs for which $N_1'N_2 = d\,11'$ where d is an integer. This condition implies that each row has d treatments in common with each column. For variance balanced designs $F^- = \varphi I$ where φ is a scalar. The dispersion matrices for this case are given in the third column of Table 1 where coefficients of the various terms have been omitted since they do not affect the objective. It can then be verified that the class of variance balanced designs with $N_1'N_2 = d\,11'$ are robust.

If $d = r$ then the designs has the property of adjusted orthogonality introduced by Eccleston and Russell (1975, 1977). These constitute an important class of designs. Raghavarao and Shah (1980) gave some such designs. Recently John and Eccleston (1986) introduced row-column α-designs having the property of adjusted orthogonality. Clearly, all such variance balanced designs are robust.

Table 1
Dispersion matrices (excluding coefficients) when row-column classification is orthogonal

$(\cdot)$	$D(\cdot)$	$D(\cdot)$ for the balanced case
$\hat{\mu}1+\hat{\rho}$	$\frac{1}{k}\left(I+\frac{1}{k}N_1'F^-N_1\right)$	N_1N_1'
$\hat{\gamma}$	$\frac{1}{h}\left(I+\frac{1}{h}N_2'F^-N_2-\frac{2}{n}11'\right)$	N_2N_2'
$\hat{\tau}$	F^-	I
$\hat{\mu}1+\hat{\rho}$, $\hat{\gamma}$	$\frac{1}{kh}N_1'F^-N_2$	N_1N_2'
$\hat{\mu}1+\hat{\rho}$, $\hat{\tau}$	$-\frac{1}{k}N_1'F^-$	N_1'
$\hat{\gamma}$, $\hat{\tau}$	$-\frac{1}{h}N_2'F^-$	N_2'

Kshirsagar's (1957) design in 9 treatments, $r=4$, $k=h=6$, all Latin square and Youden square designs, and extended Youden square designs of Shrikhande (1951) are robust since they all are variance balanced and satisfy the property of adjusted orthogonality.

4. Designs with balanced column-row classification

Since the row-column classification may not be orthogonal always, in this section the robustness of those designs is studied in which the column vs row classification is variance balanced. A generalized inverse of Z may then be chosen as φI where φ is a scalar. The various components of (2.4) are given in the second column of Table 2. As before, coefficients of the various terms have been omitted without any loss of generality. The matrices appearing there are defined below.

$$\begin{aligned}
A_1 &= I+W\{I+N_2'F^-(N_2-N_1W)-W'(I-N_1'F^-N_1)W\}W' \\
&\quad +(I+WW')N_1'F^-N_1-WN_2'F^-N_1, \\
A_2 &= I+W'(I+N_1'F^-N_1)W+N_2'F^-(N_2-N_1W), \\
A_3 &= WW'\{W+N_1'F^-(N_2-N_1W)\}-W\{I+N_2'F^-(N_2-N_1W)+N'F^-(N_2-N_1W), \\
A_4 &= WN_2'F^- -(I+WW')N_1'F^-, \qquad A_5=W'N_1'F^- -N_2'F^-.
\end{aligned}$$

In general, the robustness of a variance balanced row-column design with balanced column-row classification will need to be established by computing the matrices A_i. Two known series of robust designs are identified below.

Table 2
Dispersion matrices (excluding coefficients) when column vs row classification is balanced

$(\cdot)$	$D(\cdot)$	$D_1(\cdot)$	$D_2(\cdot)$	$D_3(\cdot)$
$\hat{\mu}1+\hat{\rho}$	A_1	$I+11'$	$N'N$	$I+11'+N_1N_1'$
$\hat{\gamma}$	A_2	$I+11'$	$I+11'$	$I+11'$
$\hat{\tau}$	F^-	I	I	I
$\hat{\mu}1+\hat{\rho}$, $\hat{\gamma}$	A_3	$11'$	N'	$11'+N_1'$
$\hat{\mu}1+\hat{\rho}$, $\hat{\tau}$	A_4	0	N'	$11'+N_1'$
$\hat{\gamma}$, $\hat{\tau}$	A_5	0	$I+11'$	$I+11'$

(i) A standard Latin square design with all diagonal elements missing is robust as can be verified using the dispersion matrices $D_1(\cdot)$ given in Table 2. For this design $N_1 = N_2 = W = 11' - I$.

(ii) Consider the series of designs obtained using Method 2 of Agrawal (1966) for which $N_1 = 11' - N$, $N_2 = 11' - I$ and $W = N'$, where N is the incidence matrix of the BIB design having parameters $v = 2k$, $b = 2(2k-1)$, $r = v - 1$, k, $\lambda = k - 1$. The various dispersion matrices $D_2(\cdot)$ for this series of designs are given in Table 2. Observe that since $N_2 = 11' - I$, only the off diagonal elements of the matrix A_5 will appear in the expression (3.4). It can then be verified that the designs in this class are robust.

A new series of variance balanced designs is defined in the following theorem. The designs of this series are robust also as discussed below.

Theorem. *If the treatment-row, treatment-column and column-row classifications in a row-column design are variance balanced such that $N_1 = W'$ and $N_2 = 11' - I$, then the design is variance balanced.*

Corollary. *A necessary condition for the existence of a design of the theorem is that a* BIB *design having parameters v, $b = p$, $r = v - 1$, k, $\lambda = k - 1$ should exist.*

The dispersion matrices $D_3(\cdot)$ for the series of designs given in the theorem are also presented in Table 2. Again, since $N_2 = 11' - I$, we only need to consider the off-diagonal elements of A_5. It can thus be verified that the designs in this class are robust.

Example. The following 12×9 balanced design in 9 treatments, r = 8, k = 6, $h = 8$ which belongs to the series of the theorem is robust. The design estimates each normalized contrast with a variance $0.1389\ \sigma^2$.

–	3	2	6	–	4	8	7	–
3	–	1	7	8	–	4	5	–
2	1	–	5	4	8	–	6	–
5	6	7	–	1	2	3	–	–
–	–	5	–	3	7	6	9	8
6	–	–	8	–	1	9	4	7
7	8	–	–	9	–	1	2	5
8	9	6	–	–	3	–	1	2
4	7	9	1	–	–	2	–	3
–	5	8	9	2	–	–	3	4
9	–	4	3	6	5	–	–	1
–	4	–	2	7	9	5	–	6

References

Agrawal, H.L. (1966). Some methods of construction of designs for two-way elimination of heterogeneity. I, *J. Amer. Statist. Assoc.* **61**, 1153–171.

Box, G.E.P. and N.R. Draper (1975), Robust designs, *Biometrika* **62**, 347–352.

Eccleston, J.A. and K.G. Russell (1975), Connectedness and orthogonality in multifactor designs, *Biometrika* **62**, 341–345.

Eccleston, J.A. and K.G. Russell (1977), Adjusted orthogonality in nonorthogonal designs, *Biometrika* **64**, 339–345.

Gopalan, R. and A. Dey (1976), On robust experimental design, *Sankhya B* **38**, 297–299.

John, J.A. and J.A. Eccleston (1986), Row-column α-designs, *Biometrika* **73**, 301–306.

Kshirsagar, A.M. (1957), On balancing in designs in which heterogeneity is eliminated in two-directions, *Calcutta Statist. Assoc. Bull.* **7**, 161–166.

Raghavarao, D. and K.R. Shah (1980), A class of D_0 designs for two-way elimination of heterogeneity, *Comm. Statist. A* **9**, 75–80.

Shrikhande, S.S. (1951), Designs for two-way elimination of heterogeneity, *Ann. Math. Statist.* **22**, 235–247.